

# HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY $(\alpha, m)$ -CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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**ABSTRACT.** In this paper, some Hermite-Hadamard type inequalities are established for harmonically  $(\alpha, m)$ -convex functions via fractional integrals and some Hermite-Hadamard type inequalities are obtained for these classes of functions.

## 1. INTRODUCTION

Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a convex function defined on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . The following inequality is well known in the literature as Hermite-Hadamard integral inequality for convex functions

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

Both inequalities hold in the reversed direction if  $f$  is concave. Note that, some of the classical inequalities for means can be obtained from appropriate particular selections of the mapping  $f$ . For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent paper [1]-[5] and references therein.

In [1], İşcan gave definition of harmonically convex functions and established some Hermite-Hadamard type inequalities for harmonically convex functions as follows:

**Definition 1.** Let  $I \subset \mathbb{R} \setminus \{0\}$  be a real interval. A function  $f : I \rightarrow \mathbb{R}$  is said to be harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \quad (1.2)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If the inequality in (1.2) is reversed, then  $f$  is said to be harmonically concave.

**Theorem 1.** [1]. Let  $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$  then the following inequalities hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.$$

**Theorem 2.** [1]. Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b \in I$  with  $a < b$ , and  $f' \in L[a, b]$ . If  $|f'|^q$  is harmonically convex on  $[a, b]$  for  $q \geq 1$ , then

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right|$$

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$$\leq \frac{ab(b-a)}{2} \lambda_1^{1-1/q} [\lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q]^{1/q}, \quad (1.3)$$

where

$$\begin{aligned} \lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right), \\ \lambda_2 &= \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right), \\ \lambda_3 &= \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right) = \lambda_1 - \lambda_2. \end{aligned}$$

**Theorem 3.** [1]. Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b \in I$  with  $a < b$ , and  $f' \in L[a, b]$ . If  $|f'|^q$  is harmonically convex on  $[a, b]$  for  $q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right)^{1/p} [\mu_1 |f'(a)|^q + \mu_2 |f'(b)|^q]^{1/q}, \end{aligned} \quad (1.4)$$

where

$$\begin{aligned} \mu_1 &= \frac{[a^{2-2q} + b^{1-2q} [(b-a)(1-2q) - a]]}{2(b-a)^2(1-q)(1-2q)}, \\ \mu_2 &= \frac{[b^{2-2q} - a^{1-2q} [(b-a)(1-2q) + b]]}{2(b-a)^2(1-q)(1-2q)}. \end{aligned}$$

In [8], Mihaşen gave definition of  $(\alpha, m)$ -convex functions as follows:

**Definition 2.** The function  $f : [0, b] \rightarrow \mathbb{R}$ ,  $b > 0$ , is said to be  $(\alpha, m)$ -convex where  $(\alpha, m) \in [0, 1]^2$ , if we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha) f(y)$$

for all  $x, y \in [0, b]$  and  $t \in [0, 1]$ .

It can be easily that for  $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$  one obtains the following classes of functions: increasing,  $\alpha$ -starshaped, starshaped,  $m$ -convex, convex,  $\alpha$ -convex.

For recent results and generalizations concerning  $(\alpha, m)$ -convex functions we refer the reader to paper [8]-[12] and references therein.

In [6], İşcan gave definition of harmonically  $(\alpha, m)$ -convex functions as follows:

**Definition 3.** The function  $f : (0, b^*] \rightarrow \mathbb{R}$ ,  $b^* > 0$ , is said to be harmonically  $(\alpha, m)$ -convex, where  $\alpha \in [0, 1]$  and  $m \in (0, 1]$ , if

$$f\left(\frac{mxy}{mty + (1-t)x}\right) \leq t^\alpha f(x) + m(1-t^\alpha) f(y) \quad (1.5)$$

for all  $x, y \in (0, b^*]$  and  $t \in [0, 1]$ . If the inequality in (1.5) is reversed, then  $f$  is said to be harmonically  $(\alpha, m)$ -concave.

Note that  $(\alpha, m) \in \{(1, m), (1, 1), (\alpha, 1)\}$  one obtains the following classes of functions: harmonically  $m$ -convex, harmonically convex, harmonically  $\alpha$ -convex (or harmonically  $s$ -convex in the first sense, if we take  $s$  instead of  $\alpha$ ).

We recall the following inequality and special functions which are known as Beta and hypergeometric function respectively

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0,$$

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \\ c &> b > 0, |z| < 1 \text{ (see [13])}. \end{aligned}$$

**Lemma 1.** [14], [15]. For  $0 < \theta \leq 1$  and  $0 \leq a < b$  we have

$$|a^\theta - b^\theta| \leq (b-a)^\theta.$$

Following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

**Definition 4.** Let  $f \in L[a, b]$ . The Riemann-Liouville integrals  $J_{a+}^\theta f$  and  $J_{b-}^\theta f$  of order  $\theta > 0$  with  $a \geq 0$  are defined by

$$J_{a+}^\theta f(x) = \frac{1}{\Gamma(\theta)} \int_a^x (x-t)^{\theta-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\theta f(x) = \frac{1}{\Gamma(\theta)} \int_x^b (t-x)^{\theta-1} f(t) dt, \quad x < b$$

respectively, where  $\Gamma$  is the Euler Gamma function defined by  $\Gamma(\theta) = \int_0^\infty e^{-t} t^{\theta-1} dt$  and  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ .

Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ , throughout this paper we will take

$$\begin{aligned} I_f(g; \theta, a, b) &= \frac{f(a) + f(b)}{2} - \frac{\Gamma(\theta+1)}{2} \left( \frac{ab}{b-a} \right)^\theta \\ &\quad \times \left\{ J_{1/a-}^\theta (f \circ g)(1/b) + J_{1/b+}^\theta (f \circ g)(1/a) \right\}. \end{aligned}$$

where  $a, b \in I$  with  $a < b$ ,  $\theta > 0$ ,  $g(x) = 1/x$ .

In [7], the authors represented Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral forms as follows:

**Theorem 4.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a function such that  $f \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $f$  is a harmonically convex function on  $[a, b]$ , then the following inequalities for fractional integrals hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\theta+1)}{2} \left( \frac{ab}{b-a} \right)^\theta \left\{ \begin{aligned} &J_{1/a-}^\theta (f \circ g)(1/b) \\ &+ J_{1/b+}^\theta (f \circ g)(1/a) \end{aligned} \right\} \leq \frac{f(a) + f(b)}{2}$$

with  $\theta > 0$ .

In [7], the authors gave the following identity for differentiable functions.

**Lemma 2.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . Then the following equality for fractional integrals holds:

$$I_f(g; \theta, a, b) = \frac{ab(b-a)}{2} \int_0^1 \frac{t^\theta - (1-t)^\theta}{(ta + (1-t)b)^2} f'\left(\frac{ab}{ta + (1-t)b}\right) dt \quad (1.6)$$

**Remark 1.** The identity (1.6) is equal the following one

$$I_f(g; \theta, a, b) = \frac{ab(b-a)}{2} \int_0^1 \frac{(1-t)^\theta - t^\theta}{(tb + (1-t)a)^2} f' \left( \frac{ab}{tb + (1-t)a} \right) dt. \quad (1.7)$$

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals. Recent results for this area, we refer the reader to paper [7], [15]-[18] and references therein.

In this paper, we aimed to establish Hermite-Hadamard's inequalities for harmonically  $(\alpha, m)$ -convex functions via fractional integrals. These results have some relations with [1].

## 2. MAIN RESULTS

**Theorem 5.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b/m \in I^\circ$  with  $a < b$ ,  $m \in (0, 1]$  and  $f' \in L[a, b]$ . If  $|f'|^q$  is harmonically  $(\alpha, m)$ -convex on  $[a, b/m]$  for some fixed  $q \geq 1$ , with  $\alpha \in [0, 1]$ , then

$$\begin{aligned} |I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\theta; a, b) \\ &\times [C_2(\theta; \alpha; a, b) |f'(a)|^q + m C_3(\theta; \alpha; a, b) |f'(b/m)|^q]^{1/q} \end{aligned} \quad (2.1)$$

where

$$C_1(\theta; a, b) = \frac{b^{-2}}{\theta+1} \left[ {}_2F_1\left(2, \theta+1; \theta+2; 1-\frac{a}{b}\right) + {}_2F_1\left(2, 1; \theta+2; 1-\frac{a}{b}\right) \right],$$

$$C_2(\theta; \alpha; a, b) = \left[ \frac{\frac{\beta(\theta+1, \alpha+1)}{b^2}}{+\frac{b^{-2}}{\theta+\alpha+1}} {}_2F_1\left(2, \theta+1; \theta+\alpha+2; 1-\frac{a}{b}\right) + {}_2F_1\left(2, 1; \theta+\alpha+2; 1-\frac{a}{b}\right) \right],$$

$$C_3(\theta; \alpha; a, b) = C_1(\theta; a, b) - C_2(\theta; \alpha; a, b).$$

*Proof.* Let  $A_t = tb + (1-t)a$ ,  $B_u = ua + (1-u)b$ . Since  $|f'|^q$  is harmonically  $(\alpha, m)$ -convex, using (1.5)

$$\begin{aligned} \left| f' \left( \frac{ab}{A_t} \right) \right|^q &= \left| f' \left( \frac{ab}{tb + (1-t)a} \right) \right|^q = \left| f' \left( \frac{ma(b/m)}{mt(b/m) + (1-t)a} \right) \right|^q \\ &\leq t^\alpha |f'(a)|^q + m(1-t^\alpha) |f'(b/m)|^q. \end{aligned} \quad (2.2)$$

From (1.7), using the property of the modulus, the power mean inequality and (2.2), we find

$$\begin{aligned}
|I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \\
&\leq \frac{ab(b-a)}{2} \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \\
&\leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} dt \right)^{1-1/q} \\
&\quad \times \left( \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \\
&\leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} dt \right)^{1-1/q} \\
&\quad \times \left( \begin{aligned} &\left( \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} t^\alpha dt \right) |f'(a)|^q \\ &+ m \left( \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} (1-t^\alpha) dt \right) |f'(b/m)|^q \end{aligned} \right)^{1/q} \quad (2.3)
\end{aligned}$$

calculating following integrals, we have

$$\begin{aligned}
\int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} dt &= \int_0^1 \frac{u^\theta + (1-u)^\theta}{B_u^2} du \\
&= \frac{b^{-2}}{\theta+1} \left[ {}_2F_1 \left( 2, \theta+1; \theta+2; 1-\frac{a}{b} \right) \right. \\
&\quad \left. + {}_2F_1 \left( 2, 1; \theta+2; 1-\frac{a}{b} \right) \right] \\
&= C_1(\theta; a, b) \quad (2.4)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} t^\alpha dt &= \int_0^1 \frac{u^\theta + (1-u)^\theta}{B_u^2} (1-u)^\alpha du \\
&= \left[ \frac{\frac{\beta(\theta+1, \alpha+1)}{b^2}}{+\frac{b^{-2}}{\theta+\alpha+1}} {}_2F_1 \left( 2, \theta+1; \theta+\alpha+2; 1-\frac{a}{b} \right) \right. \\
&\quad \left. + {}_2F_1 \left( 2, 1; \theta+\alpha+2; 1-\frac{a}{b} \right) \right] \\
&= C_2(\theta; \alpha; a, b) \quad (2.5)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} (1-t^\alpha) dt &= \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} dt - \int_0^1 \frac{(1-t)^\theta + t^\theta}{A_t^2} t^\alpha dt \\
&= C_1(\theta; a, b) - C_2(\theta; \alpha; a, b) \\
&= C_3(\theta; \alpha; a, b) \quad (2.6)
\end{aligned}$$

Thus, if we use (2.4)-(2.6) in (2.3) we get the inequality of (2.1) and this completes the proof.  $\square$

**Corollary 1.** *In Theorem 5,*

- (a) *If we take  $\alpha = 1$ ,  $m = 1$  we have the following inequality for harmonically convex functions:*

$$\begin{aligned}
|I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\theta; a, b) \\
&\quad \times [C_2(\theta; 1; a, b) |f'(a)|^q + C_3(\theta; 1; a, b) |f'(b)|^q]^{1/q},
\end{aligned}$$

(b) If we take  $\alpha = 1$  we have the following inequality for harmonically  $m$ -convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\theta; a, b) \times [C_2(\theta; 1; a, b) |f'(a)|^q + mC_3(\theta; 1; a, b) |f'(b/m)|^q]^{1/q},$$

(c) If we take  $m = 1$  we have the following inequality for harmonically  $\alpha$ -convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\theta; a, b) \times [C_2(\theta; \alpha; a, b) |f'(a)|^q + C_3(\theta; \alpha; a, b) |f'(b)|^q]^{1/q}.$$

When  $0 < \theta \leq 1$ , using Lemma 1 we shall give another result for harmonically  $(\alpha, m)$ -convex functions as follows:

**Theorem 6.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b/m \in I^\circ$  with  $a < b$ ,  $m \in (0, 1]$  and  $f' \in L[a, b]$ . If  $|f'|^q$  is harmonically  $(\alpha, m)$ -convex on  $[a, b/m]$  for some fixed  $q \geq 1$ , with  $\alpha \in [0, 1]$ , then

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2} C_4^{1-1/q}(\theta; a, b) \times [C_5(\theta; \alpha; a, b) |f'(a)|^q + mC_6(\theta; \alpha; a, b) |f'(b/m)|^q]^{1/q} \quad (2.7)$$

where  $0 < \theta \leq 1$  and

$$C_4(\theta; a, b) = \left[ \begin{array}{c} \frac{b^{-2}}{\theta+1} {}_2F_1\left(2, 1; \theta+2; 1 - \frac{a}{b}\right) \\ - \frac{b}{\theta+1} {}_2F_1\left(2, \theta+1; \theta+2; 1 - \frac{a}{b}\right) \\ + \left(\frac{a+b}{2}\right)^{-2} \frac{1}{\theta+1} {}_2F_1\left(2, \theta+1; \theta+2; \frac{b-a}{b+a}\right) \end{array} \right],$$

$$C_5(\theta; \alpha; a, b) = \left[ \begin{array}{c} \frac{b^{-2}}{\theta+\alpha+1} {}_2F_1\left(2, 1; \theta+\alpha+2; 1 - \frac{a}{b}\right) \\ - \frac{\beta(\theta+1, \alpha+1)}{b^2} {}_2F_1\left(2, \theta+1; \theta+\alpha+2; 1 - \frac{a}{b}\right) \\ + \frac{\beta(\theta+1, \alpha+1)}{(a+b)^2 2^{\alpha-2}} {}_2F_1\left(2, \theta+1; \theta+\alpha+2; \frac{b-a}{b+a}\right) \end{array} \right],$$

$$C_6(\theta; \alpha; a, b) = C_4(\theta; a, b) - C_5(\theta; \alpha; a, b).$$

*Proof.* Let  $A_t = tb + (1-t)a$ ,  $B_u = ua + (1-u)b$ . From (1.7), using the power mean inequality and (2.2), we find

$$\begin{aligned}
|I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \\
&\leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} dt \right)^{1-1/q} \\
&\quad \times \left( \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \\
&\leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} dt \right)^{1-1/q} \\
&\quad \times \left( \begin{aligned} &\left( \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} t^\alpha dt \right) |f'(a)|^q \\ &+ m \left( \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} (1-t^\alpha) dt \right) |f'(b/m)|^q \end{aligned} \right)^{1/q} \quad (2.8)
\end{aligned}$$

calculating following integrals by Lemma 1, we have

$$\begin{aligned}
\int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} dt &= \int_0^{1/2} \frac{(1-t)^\theta - t^\theta}{A_t^2} dt + \int_{1/2}^1 \frac{t^\theta - (1-t)^\theta}{A_t^2} dt \\
&= \int_0^1 \frac{t^\theta - (1-t)^\theta}{A_t^2} dt + 2 \int_0^{1/2} \frac{(1-t)^\theta - t^\theta}{A_t^2} dt \\
&\leq \int_0^1 \frac{t^\theta}{A_t^2} dt - \int_0^1 \frac{(1-t)^\theta}{A_t^2} dt + 2 \int_0^{1/2} \frac{(1-2t)^\theta}{A_t^2} dt \\
&= \int_0^1 \frac{(1-u)^\theta}{B_u^2} du - \int_0^1 \frac{u^\theta}{B_u^2} du + \int_0^1 \frac{(1-u)^\theta}{(\frac{u}{2}b + (1-\frac{u}{2})a)^2} du \\
&= \int_0^1 \frac{(1-u)^\theta}{B_u^2} du - \int_0^1 \frac{u^\theta}{B_u^2} du \\
&\quad + \int_0^1 v^\theta \left( \frac{a+b}{2} \right)^{-2} \left( 1 - v \left( \frac{b-a}{b+a} \right) \right)^{-2} dv \\
&= \left[ \begin{aligned} &\frac{b^{-2}}{\theta+1} {}_2F_1 \left( 2, 1; \theta+2; 1 - \frac{a}{b} \right) \\ &- \frac{b^{-2}}{\theta+1} {}_2F_1 \left( 2, \theta+1; \theta+2; 1 - \frac{a}{b} \right) \\ &+ \left( \frac{a+b}{2} \right)^{-2} \frac{1}{\theta+1} {}_2F_1 \left( 2, \theta+1; \theta+2; \frac{b-a}{b+a} \right) \end{aligned} \right] \\
&= C_4(\theta; a, b) \quad (2.9)
\end{aligned}$$

and similarly we get

$$\begin{aligned}
\int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} t^\alpha dt &\leq \int_0^1 \frac{t^{\theta+\alpha}}{A_t^2} dt - \int_0^1 \frac{(1-t)^\theta t^\alpha}{A_t^2} dt + 2 \int_0^{1/2} \frac{(1-2t)^\theta t^\alpha}{A_t^2} dt \\
&= \int_0^1 \frac{(1-u)^{\theta+\alpha}}{B_u^2} du - \int_0^1 \frac{u^\theta (1-u)^\alpha}{B_u^2} du \\
&\quad + \int_0^1 \frac{(1-u)^\theta (\frac{u}{2})^\alpha}{(\frac{u}{2}b + (1-\frac{u}{2})a)^2} du \\
&= \int_0^1 \frac{(1-u)^{\theta+\alpha}}{B_u^2} du - \int_0^1 \frac{u^\theta (1-u)^\alpha}{B_u^2} du \\
&\quad + \frac{(\frac{a+b}{2})^{-2}}{2^\alpha} \int_0^1 v^\theta (1-v)^\alpha \left(1-v \left(\frac{b-a}{b+a}\right)\right)^{-2} dv \\
&= \left[ \begin{aligned} &\frac{b^{-2}}{\theta+\alpha+1} {}_2F_1\left(2, 1; \theta+\alpha+2; 1-\frac{a}{b}\right) \\ &- \frac{\beta(\theta+1, \alpha+1)}{b^2} {}_2F_1\left(2, \theta+1; \theta+\alpha+2; 1-\frac{a}{b}\right) \\ &+ \frac{\beta(\theta+1, \alpha+1)}{(a+b)^2 2^{\alpha-2}} {}_2F_1\left(2, \theta+1; \theta+\alpha+2; \frac{b-a}{b+a}\right) \end{aligned} \right] \\
&= C_5(\theta; \alpha; a, b) \tag{2.10}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} (1-t^\alpha) dt &= \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} dt - \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} t^\alpha dt \\
&= C_4(\theta; a, b) - C_5(\theta; \alpha; a, b) \\
&= C_6(\theta; \alpha; a, b) \tag{2.11}
\end{aligned}$$

Thus, if we use (2.9)-(2.11) in (2.8) we get the inequality of (2.7) and this completes the proof.  $\square$

**Remark 2.** If we take  $\theta = 1$ ,  $\alpha = 1$ ,  $m = 1$  in Theorem 6, then inequality (2.7) becomes inequality (1.3) of Theorem 2.

**Corollary 2.** In Theorem 6,

- (a) If we take  $\alpha = 1$ ,  $m = 1$  we have the following inequality for harmonically convex functions:

$$\begin{aligned}
|I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2} C_4^{1-1/q}(\theta; a, b) \\
&\quad \times [C_5(\theta; 1; a, b) |f'(a)|^q + C_6(\theta; 1; a, b) |f'(b)|^q]^{1/q},
\end{aligned}$$

- (b) If we take  $\alpha = 1$  we have the following inequality for harmonically  $m$ -convex functions:

$$\begin{aligned}
|I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2} C_4^{1-1/q}(\theta; a, b) \\
&\quad \times [C_5(\theta; 1; a, b) |f'(a)|^q + m C_6(\theta; 1; a, b) |f'(b/m)|^q]^{1/q},
\end{aligned}$$

- (c) If we take  $m = 1$  we have the following inequality for harmonically  $\alpha$ -convex functions:

$$\begin{aligned}
|I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2} C_4^{1-1/q}(\theta; a, b) \\
&\quad \times [C_5(\theta; \alpha; a, b) |f'(a)|^q + C_6(\theta; \alpha; a, b) |f'(b)|^q]^{1/q}.
\end{aligned}$$



**Theorem 7.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b/m \in I^\circ$  with  $a < b$ ,  $m \in (0, 1]$  and  $f' \in L[a, b]$ . If  $|f'|^q$  is harmonically  $(\alpha, m)$ -convex on  $[a, b/m]$  for some fixed  $q > 1$ , with  $\alpha \in [0, 1]$ , then

$$|I_f(g; \theta, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + m\alpha |f'(b/m)|^q}{\alpha + 1} \right)^{1/q} \\ \times \left[ {}_2F_1^{1/p} \left( 2p, \theta p + 1; \theta p + 2; 1 - \frac{a}{b} \right) + {}_2F_1^{1/p} \left( 2p, 1; \theta p + 2; 1 - \frac{a}{b} \right) \right] \quad (2.12)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* Let  $A_t = tb + (1-t)a$ ,  $B_u = ua + (1-u)b$ . From (1.7), using the property of the modulus, the Hölder inequality and (2.2), we find

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2} \left[ \int_0^1 \frac{(1-t)^\theta}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right| dt + \int_0^1 \frac{t^\theta}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \right] \\ \leq \frac{ab(b-a)}{2} \left[ \left( \int_0^1 \frac{(1-t)^{\theta p}}{A_t^{2p}} dt \right)^{1/p} \left( \int_0^1 \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} + \left( \int_0^1 \frac{t^{\theta p}}{A_t^{2p}} dt \right)^{1/p} \left( \int_0^1 \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \right] \\ \leq \frac{ab(b-a)}{2} \left( \left( \int_0^1 \frac{u^{\theta p}}{B_u^{2p}} du \right)^{1/p} + \left( \int_0^1 \frac{(1-u)^{\theta p}}{B_u^{2p}} du \right)^{1/p} \right) \\ \times \left( \int_0^1 t^\alpha |f'(a)|^q + m(1-t^\alpha) |f'(b/m)|^q dt \right)^{1/q} \\ = \frac{ab(b-a)}{2} \left( K_1^{1/p} + K_2^{1/p} \right) \left( \frac{|f'(a)|^q + m\alpha |f'(b/m)|^q}{\alpha + 1} \right)^{1/q} \quad (2.13)$$

calculating  $K_1$  and  $K_2$  we have

$$K_1 = \int_0^1 \frac{u^{\theta p}}{B_u^{2p}} du = \frac{b^{-2p}}{\theta p + 1} {}_2F_1 \left( 2p, \theta p + 1; \theta p + 2; 1 - \frac{a}{b} \right) \quad (2.14)$$

$$K_2 = \int_0^1 \frac{(1-u)^{\theta p}}{B_u^{2p}} du = \frac{b^{-2p}}{\theta p + 1} {}_2F_1 \left( 2p, 1; \theta p + 2; 1 - \frac{a}{b} \right) \quad (2.15)$$

Thus, if we use (2.14), (2.15) in (2.13) we get the inequality of (2.12) and this

completes the proof.  $\square$

**Corollary 3.** In Theorem 7,

- (a) If we take  $\alpha = 1$ ,  $m = 1$  we have the following inequality for harmonically convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q} \\ \times \left[ {}_2F_1^{1/p} \left( 2p, \theta p + 1; \theta p + 2; 1 - \frac{a}{b} \right) + {}_2F_1^{1/p} \left( 2p, 1; \theta p + 2; 1 - \frac{a}{b} \right) \right],$$

(b) If we take  $\alpha = 1$  we have the following inequality for harmonically  $m$ -convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + m|f'(b/m)|^q}{2} \right)^{1/q} \\ \times \left[ {}_2F_1^{1/p} \left( 2p, \theta p + 1; \theta p + 2; 1 - \frac{a}{b} \right) + {}_2F_1^{1/p} \left( 2p, 1; \theta p + 2; 1 - \frac{a}{b} \right) \right],$$

(c) If we take  $m = 1$  we have the following inequality for harmonically  $\alpha$ -convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + \alpha|f'(b)|^q}{\alpha + 1} \right)^{1/q} \\ \times \left[ {}_2F_1^{1/p} \left( 2p, \theta p + 1; \theta p + 2; 1 - \frac{a}{b} \right) + {}_2F_1^{1/p} \left( 2p, 1; \theta p + 2; 1 - \frac{a}{b} \right) \right].$$

**Theorem 8.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b/m \in I^\circ$  with  $a < b$ ,  $m \in (0, 1]$  and  $f' \in L[a, b]$ . If  $|f'|^q$  is harmonically  $(\alpha, m)$ -convex on  $[a, b/m]$  for some fixed  $q > 1$ , with  $\alpha \in [0, 1]$ , then

$$|I_f(g; \theta, a, b)| \leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{1}{\alpha + 1} \right)^{1/q} \\ \times \left( {}_2F_1 \left( 2q, 1; \alpha + 2; 1 - \frac{a}{b} \right) |f'(a)|^q + m \left[ {}_2F_1 \left( 2q, 1; 2; 1 - \frac{a}{b} \right) - {}_2F_1 \left( 2q, 1; \alpha + 2; 1 - \frac{a}{b} \right) \right] |f'(b/m)|^q \right)^{1/q} \quad (2.16)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* Let  $A_t = tb + (1-t)a$ ,  $B_u = ua + (1-u)b$ . From (1.7), using the Hölder inequality, Lemma 1, and (2.2), we find

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \\ \leq \frac{ab(b-a)}{2} \left( \int_0^1 |(1-t)^\theta - t^\theta|^p dt \right)^{1/p} \\ \times \left( \int_0^1 \frac{1}{A_t^{2q}} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \\ \leq \frac{ab(b-a)}{2} \left( \int_0^1 |1-2t|^{\theta p} dt \right)^{1/p} \\ \times \left( \int_0^1 \frac{1}{A_t^{2q}} [t^\alpha |f'(a)|^q + m(1-t^\alpha) |f'(b/m)|^q] dt \right)^{1/q} \quad (2.17)$$

calculating following integrals, we have

$$\int_0^1 |1-2t|^{\theta p} dt = \frac{1}{\theta p + 1} \quad (2.18)$$

$$\int_0^1 \frac{t^\alpha}{A_t^{2q}} dt = \int_0^1 \frac{(1-u)^\alpha}{B_u^{2q}} dt = \frac{b^{-2q}}{\alpha + 1} {}_2F_1 \left( 2q, 1; \alpha + 2; 1 - \frac{a}{b} \right) \quad (2.19)$$

$$\begin{aligned} \int_0^1 \frac{1-t^\alpha}{A_t^{2q}} dt &= \int_0^1 \frac{1-(1-u)^\alpha}{B_u^{2q}} dt = b^{-2q} {}_2F_1\left(2q, 1; 2; 1 - \frac{a}{b}\right) \\ &\quad - \frac{b^{-2q}}{\alpha+1} {}_2F_1\left(2q, 1; \alpha+2; 1 - \frac{a}{b}\right) \end{aligned} \quad (2.20)$$

Thus, if we use (2.18)-(2.20) in (2.17) we get the inequality of (2.16) and this completes the proof.  $\square$

**Remark 3.** If we take  $\theta = 1$ ,  $\alpha = 1$ ,  $m = 1$  in Theorem 8, then inequality (2.16) becomes inequality (1.4) of Theorem 3.

**Corollary 4.** In Theorem 8,

- (a) If we take  $\alpha = 1$ ,  $m = 1$  we have the following inequality for harmonically convex functions:

$$\begin{aligned} |I_f(g; \theta, a, b)| &\leq \frac{a(b-a)}{2^{1+1/q}b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \\ &\quad \times \left( \begin{array}{c} {}_2F_1\left(2q, 1; 3; 1 - \frac{a}{b}\right) |f'(a)|^q \\ 2 {}_2F_1\left(2q, 1; 2; 1 - \frac{a}{b}\right) \\ - {}_2F_1\left(2q, 1; 3; 1 - \frac{a}{b}\right) \end{array} \left[ |f'(b)|^q \right] \right)^{1/q}, \end{aligned}$$

- (b) If we take  $\alpha = 1$  we have the following inequality for harmonically  $m$ -convex functions:

$$\begin{aligned} |I_f(g; \theta, a, b)| &\leq \frac{a(b-a)}{2^{1+1/q}b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \\ &\quad \times \left( \begin{array}{c} {}_2F_1\left(2q, 1; 3; 1 - \frac{a}{b}\right) |f'(a)|^q \\ 2 {}_2F_1\left(2q, 1; 2; 1 - \frac{a}{b}\right) \\ - {}_2F_1\left(2q, 1; 3; 1 - \frac{a}{b}\right) \end{array} \left[ |f'(b/m)|^q \right] \right)^{1/q}, \end{aligned}$$

- (c) If we take  $m = 1$  we have the following inequality for harmonically  $\alpha$ -convex functions:

$$\begin{aligned} |I_f(g; \theta, a, b)| &\leq \frac{a(b-a)}{2b} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{1}{\alpha + 1} \right)^{1/q} \\ &\quad \times \left( \begin{array}{c} {}_2F_1\left(2q, 1; \alpha+2; 1 - \frac{a}{b}\right) |f'(a)|^q \\ (\alpha+1) {}_2F_1\left(2q, 1; 2; 1 - \frac{a}{b}\right) \\ - {}_2F_1\left(2q, 1; \alpha+2; 1 - \frac{a}{b}\right) \end{array} \left[ |f'(b)|^q \right] \right)^{1/q}. \end{aligned}$$

**Theorem 9.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b/m \in I^\circ$  with  $a < b$ ,  $m \in (0, 1]$  and  $f' \in L[a, b]$ . If  $|f'|^q$  is harmonically  $(\alpha, m)$ -convex on  $[a, b/m]$  for some fixed  $q > 1$ , with  $\alpha \in [0, 1]$ , then

$$\begin{aligned} |I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2^{(1/p)-1}(a+b)^2} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + m\alpha |f'(b/m)|^q}{\alpha + 1} \right)^{1/q} \\ &\quad \times \left[ \begin{array}{c} {}_2F_1\left(2p, \theta p + 1; \theta p + 2; \frac{b-a}{b+a}\right) \\ + {}_2F_1\left(2p, \theta p + 1; \theta p + 2; \frac{a-b}{b+a}\right) \end{array} \right]^{1/p} \end{aligned} \quad (2.21)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* Let  $A_t = tb + (1 - t)a$ . From (1.7), using the Hölder inequality, Lemma 1, and (2.2), we find

$$\begin{aligned}
|I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\theta - t^\theta|}{A_t^{2p}} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \\
&\leq \frac{ab(b-a)}{2} \int_0^1 \frac{|1-2t|^\theta}{A_t^{2p}} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \\
&\leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{|1-2t|^{\theta p}}{A_t^{2p}} dt \right)^{1/p} \left( \int_0^1 \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \\
&\leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{|1-2t|^{\theta p}}{A_t^{2p}} dt \right)^{1/p} \\
&\quad \times \left( \int_0^1 t^\alpha |f'(a)|^q + m(1-t^\alpha) |f'(b/m)|^q dt \right)^{1/q} \\
&= \frac{ab(b-a)}{2} \left[ \int_0^{1/2} \frac{(1-2t)^{\theta p}}{A_t^{2p}} dt + \int_{1/2}^1 \frac{(2t-1)^{\theta p}}{A_t^{2p}} dt \right]^{1/p} \\
&\quad \times \left( \frac{|f'(a)|^q + m\alpha |f'(b/m)|^q}{\alpha + 1} \right)^{1/q} \tag{2.22}
\end{aligned}$$

calculating following integrals, we have

$$\begin{aligned}
\int_0^{1/2} \frac{(1-2t)^{\theta p}}{A_t^{2p}} dt &= \frac{1}{2} \int_0^1 \frac{(1-u)^{\theta p}}{\left(\frac{u}{2}b + \left(1 - \frac{u}{2}\right)a\right)^{2p}} du \\
&= \frac{(a+b)^{-2p}}{2^{1-2p}} \int_0^1 v^{\theta p} \left(1 - v \left(\frac{b-a}{b+a}\right)\right)^{-2p} dv \\
&= \frac{(a+b)^{-2p}}{2^{1-2p}(\theta p + 1)} {}_2F_1\left(2p, \theta p + 1; \theta p + 2; \frac{b-a}{b+a}\right) \tag{2.23}
\end{aligned}$$

$$\begin{aligned}
\int_{1/2}^1 \frac{(2t-1)^{\theta p}}{A_t^{2p}} dt &= \frac{1}{2} \int_1^2 \frac{(u-1)^{\theta p}}{\left(\frac{u}{2}b + \left(1 - \frac{u}{2}\right)a\right)^{2p}} du \\
&= \frac{(a+b)^{-2p}}{2^{1-2p}} \int_0^1 v^{\theta p} \left(1 - v \left(\frac{a-b}{b+a}\right)\right)^{-2p} dv \\
&= \frac{(a+b)^{-2p}}{2^{1-2p}(\theta p + 1)} {}_2F_1\left(2p, \theta p + 1; \theta p + 2; \frac{a-b}{b+a}\right) \tag{2.24}
\end{aligned}$$

Thus, if we use (2.23)-(2.24) in (2.22) we get the inequality of (2.21) and this completes the proof.  $\square$

**Corollary 5.** *In Theorem 9,*

- (a) *If we take  $\alpha = 1$ ,  $m = 1$  we have the following inequality for harmonically convex functions:*

$$\begin{aligned}
|I_f(g; \theta, a, b)| &\leq \frac{ab(b-a)}{2^{(1/p)-1}(a+b)^2} \left(\frac{1}{\theta p + 1}\right)^{1/p} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q} \\
&\quad \times \left[ {}_2F_1\left(2p, \theta p + 1; \theta p + 2; \frac{b-a}{b+a}\right) + {}_2F_1\left(2p, \theta p + 1; \theta p + 2; \frac{a-b}{b+a}\right) \right]^{1/p},
\end{aligned}$$

(b) If we take  $\alpha = 1$  we have the following inequality for harmonically  $m$ -convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2^{(1/p)-1}(a+b)^2} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + m|f'(b/m)|^q}{2} \right)^{1/q} \\ \times \left[ {}_2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{b-a}{b+a} \right) + {}_2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{a-b}{b+a} \right) \right]^{1/p},$$

(c) If we take  $m = 1$  we have the following inequality for harmonically  $\alpha$ -convex functions:

$$|I_f(g; \theta, a, b)| \leq \frac{ab(b-a)}{2^{(1/p)-1}(a+b)^2} \left( \frac{1}{\theta p + 1} \right)^{1/p} \left( \frac{|f'(a)|^q + \alpha|f'(b)|^q}{\alpha + 1} \right)^{1/q} \\ \times \left[ {}_2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{b-a}{b+a} \right) + {}_2F_1 \left( 2p, \theta p + 1; \theta p + 2; \frac{a-b}{b+a} \right) \right]^{1/p}.$$

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